Case Study: Delta-Operator Formulated Digital Filters

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***Abstract*—The shift operator, q, and the z-transform are used extensively in filter design. However, techniques involving forward shifts do not have sensible continuous-time counterparts. The delta operator and its corresponding transform, alternatives to the previous methods, converge to the underlying continuos-time result as the sampling period goes to zero. This unifies continuous and discrete time theory. It has also been shown mathematically that the delta operator is numerically superior to the shift operator. This document will examine the delta operator and its transform.**

***Index Terms*—Delta operator, shift operator, high-speed DSP.**

I. Introduction

DIGITAL filter design, a widely explored and vital aspect of digital signal processing, is traditionally performed with the shift operator q and the z-transform. However, as high-speed DSP applications become more prevalent, the limitations of these methods become apparent. Given a continuous time system whose Laplace tranform is $H(s)$, the typical methods of transforming it into a digital equivalent include the impulse-invariance method, the bilinear transform, and the zero-order hold. Instead of converging to the underlying continuous-time equivalent as the sampling period goes to zero, the poles of the discrete-time filter converge to $z = 1$. Since the poles are tending towards the boundary of the unit circle, the filter becomes very sensitive to changes in the coefficients. Coefficient quantization can even lead the filter to become unstable as a pole could venture outside of the unit circle.

Clearly, a different method for obtaining a digital equivalent of a continuous time system is desirable, and this is exactly what the δ operator provides. Moreover, the δ operator leads to models with a one-to-one linear relationship with shift operator models, converges to a continuos-time derivative as the sampling period goes to zero, and has an inverse that is causal. [2] This document explores the δ operator and its corresponding transform. There will also be a discussion on finite wordlength effects and a design example.

II. The Delta Operator

This sections will explore the basics of the δ operator and some of its useful properties.

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A. Definition

We define the delta operator as a forward difference by

$$\delta \triangleq \frac{q-1}{\Delta} \tag{1}$$

where Δ is the sampling period. It can be applied on a system as follows:

$$\delta u[k] = \frac{u[k+1] - u[k]}{\Delta} \tag{2}$$

As seen from equation 2, the delta operator approximates the derivative. It can be shown that if $u[k] = u(k\Delta)$ results from sampling a continuous signal with sampling period Δ and the delta operator is applied, then the result is approximately the derivative of the continuous signal. [2]

$$\delta u[k] \approx \left. \frac{du(t)}{dt} \right|_{u=u(k\Delta)} \tag{3}$$

This approximation becomes better as $\Delta \to 0$.

The inverse delta operator, δ^{-1}, is defined by

$$\delta^{-1} u = \sum_{m=0}^{k-1} \Delta u[m] \tag{4}$$

This lower Riemann sum converges to the Riemann integral as $\Delta \to 0$. [2] This shows the close relationship the delta operator has with continuous-time theory as the sampling time goes to zero.

B. Properties

Because of this close link with continuous-time theory, the delta operator and its inverse behave in much the same way as the derivative and the integral. The following properties of the δ operator follow directly from the definitions above. [2]

$$\delta \left\{ \sum_{m=0}^{k-1} \Delta f[m] \right\} = f[k] \tag{5}$$

$$\delta \left\{ \sum_{m=k}^{L} \Delta f[m] \right\} = -f[k] \tag{6}$$

$$\sum_{m=\alpha}^{\beta} \Delta f[m] = f[\beta+1] - f[\alpha] \tag{7}$$

$$\delta(fg) = (\delta f)g + f(\delta g) + \Delta(\delta f)(\delta g) \tag{8}$$

The region of convergence of the delta operator is defined by the following equation and seen in figure 1.

$$\frac{\Delta}{2} \left| \gamma^2 + Real\{\gamma\} \right| < 0 \tag{9}$$

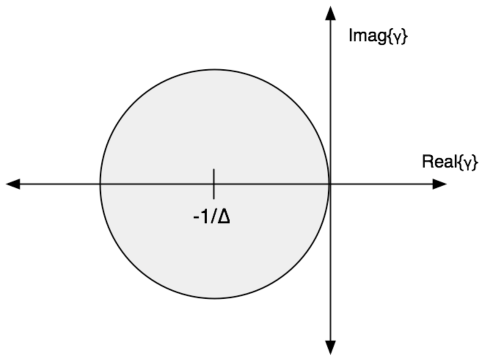


Fig. 1. Region of convergence for the delta operator

Note that as the sampling period $\Delta \rightarrow 0$, the ROC becomes $Real\{s\} < 0$ which corresponds the the left plane. [1]

III. The Discrete Delta Transform

Recall that the one-sided Laplace transform is given by

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt \tag{10}$$

This holds for continuous-time systems, but an integral has little use in discrete applications. Therefore, the delta transform will be defined as the summation of the continuous time signal $u(t)$ sampled at a period Δ.

$$Y'(s) = \sum_{k=0}^{\infty} \Delta e^{-sk\Delta} y(k\Delta) \tag{11}$$

With the above formula, s will always appear in the form of $e^{s\Delta}$. It becomes useful in solving difference equations with the delta transform to substitute this with something more appropriate. When the following substitution is used

$$e^{s\Delta} = 1 + \Delta\gamma \tag{12}$$

in the solution, δ can simply be replaced by γ. [2] In substituting (12) into (11), we arrive at the final form of the delta transform given by

$$Y_\delta(\gamma) = \mathcal{D}\{y[k]\} \triangleq \sum_{k=0}^{\infty} \Delta(1 + \Delta\gamma)^{-k} y[k] \tag{13}$$

This is a Riemann sum approximation to the Laplace transform. It would be expected that as the sampling period goes to zero that the sum would converge to the integral, and therefore the delta transform converge to the Laplace transform.

Such a result would be highly desirable because as the sampling frequency increases, accuracy actually improves. This property can be demonstrated with the classic transform example of an exponential taken from [2].

$$y(t) = e^{at}\mu(t) \tag{14}$$

where $\mu(t)$ is the unit step function. The unit-step function agrees with both the integral and summation starting at zero. Note that from equation 10 the one-sided Laplace transform of equation 14 is

$$Y(s) = \frac{1}{s-a} \tag{15}$$

The discrete, sampled form of (14) with a sampling period Δ is

$$y[k] = e^{ak\Delta}\mu(k) \tag{16}$$

And in using the definition of the delta transform from (13) on our sampled signal we arrive at

$$\begin{aligned} Y_\delta(\gamma) &= \sum_{k=0}^{\infty} \Delta(1+\gamma\Delta)^{-k} e^{ak\Delta} \\ &= \frac{\Delta}{(1-(1+\Delta\gamma)^{-1})e^{a\Delta}} \\ &= \frac{\Delta(1+\Delta\gamma)}{1+\Delta\gamma - e^{a\Delta}} \\ &= \frac{1+\Delta\gamma}{\gamma - \left(\frac{e^{a\Delta}-1}{\Delta}\right)} \end{aligned} \tag{17}$$

Therefore, we conclude that (16) and (17) form a delta transform pair. This pair can be used to solve most simple problems encountered using the delta transform. [2] Also, note that

$$\lim_{\Delta\rightarrow 0} \frac{1+\Delta\gamma}{\gamma - \left[\frac{e^{a\Delta}-1}{\Delta}\right]} = \frac{1}{\gamma - a} \tag{18}$$

This convergence to the Laplace transform is exactly what we expected. This result unifies continuous and discrete time theory.

For sake of completeness, the inverse of the delta transform is given by

$$x[k] = \frac{1}{2\pi j} \oint X_\delta(\gamma)(1+\Delta\gamma)^{k-1}\, d\gamma \tag{19}$$

where the contour of integration lies within the ROC of $X_\delta(\gamma)$ and encircles all singularities once in the anti-clockwise sense. [2]

IV. Delta Transform Properties

Table I shows some of the properties associated with the δ transform. [2] Notice the similarities between these and the properties for the Laplace transform.

Two other useful properties are the Initial Value Theorem and the Final Value Theorem given by equations 20 and 21 respectively.

$$\lim_{k\rightarrow 0} x[k] = \lim_{\gamma\rightarrow\infty} \left[\frac{\gamma X_\delta(\gamma)}{1+\Delta\gamma}\right] \tag{20}$$

$$\lim_{k\rightarrow\infty} x[k] = \lim_{\gamma\rightarrow 0} \gamma X_\delta(\gamma) \tag{21}$$

TABLE I
PROPERTIES OF THE DELTA TRANSFORM

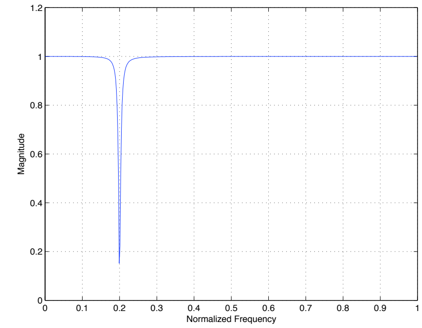
Sequence	Delta Transform
$x[k]$	$X_\delta(\gamma)$
$y[k]$	$Y_\delta(\gamma)$
$ax[k] + by[k]$	$aX_\delta(\gamma) + bY_\delta(\gamma)$
$x[k-k_0]\,\mu[k-k_0]\,, k_0 > 0$	$(1-\Delta\gamma)^{-k_0} X_\delta(\gamma)$
$x[k] \circledast y[k]$	$X_\delta(\gamma) Y_\delta(\gamma)$
$\delta x[k]$	$\gamma X_\delta(\gamma) - (1+\Delta\gamma)x[0]$
$\delta^{-1} x[k] = \sum_{m=0}^{k-1} \Delta x[m]$	$\frac{1}{\gamma} X_\delta(\gamma)$
$k\Delta x[k]$	$-(1+\Delta\gamma)\frac{d}{d\gamma} X_\delta(\gamma)$
$\frac{x[k]}{k\Delta}$	$\int_\gamma^\infty \frac{X_\delta(\eta)}{1+\Delta\eta}\, d\eta$
$(1+\Delta a)^k x[k]$	$X_\delta\left(\frac{\gamma - a}{1+\Delta a}\right)$

V. Finite Wordlength Effects

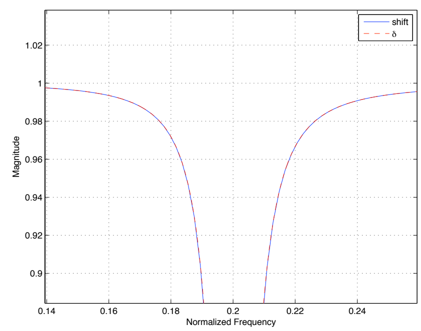
It has been shown that when the sampling time decreases and the shift operator poles tend towards $z = 1$ that the delta operator and its transform have several advantages over the shift operator and the z-transform. [6] When the poles of the transfer function are very close to the unit circle, quantization and finite wordlength issues may cause a filter to behave erratically or even become unstable if a pole ventures outside of $z = 1$.

To demonstrate this point, a notch filter with a center frequency at .2 times the nyquist frequency with a very narrow bandwidth is selected. This filter can be seen in figure 2a. To simulate the effects of finite wordlength, the coefficients of both the z-transform and δ-transform filters will be rounded to a certain number of significant digits. Figure 2b compares the two methods with five significant digits allowed in the coefficients. As we can see from the close-up, the two methods are virtually identical and properly causing a notch to occur at .2 of the nyquist frequency. However, the difference between the methods becomes apparent with less significant digits. As seen in figure 2c when only two significant digits remain, the difference between the two methods is extreme. The δ-transform filter is still keeping an appropriate magnitude shape for a notch filter and is still attenuating .2 of the nyquist. The z-transform filter, however, barely even resembles a notch filter. Its magnitude is almost at its peak when it should be at a minimum.

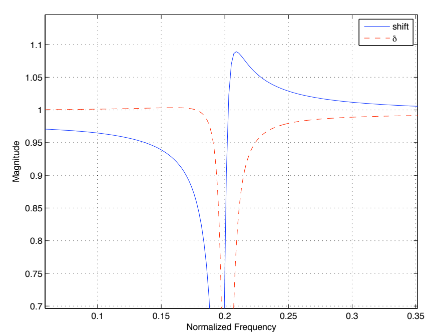
This clearly demonstrates the ability for the delta transform to perform better than the z-transform under similar finite wordlength constraints. It should be mentioned that it was shown in [4], [5] that the bandwidth of a filter must be 10 to 50 times lower that the sampling frequency for the delta operator to perform better than the shift operator.



(a)



(b)

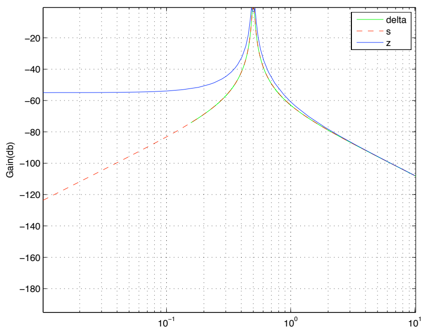


(c)

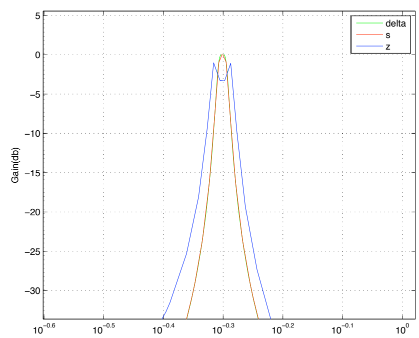
Fig. 2. (a) A notch filter with a center frequency at .2 the nyquist frequency (b) Close-up comparison between shift and delta operators when filter coefficients rounded to five significant digits (c) Close-up comparison with coefficients rounded to two significant digits.

VI. Design Example

To further demonstrate the advantages of the delta operator over the shift operator, a narrow bandpass digital filter will be created using both methods. Figure 3a shows an elliptical filter comparing the analog, z-transform, and δ-transform design methods. The sampling period used for this was quite small, and it is apparent that the z-transform method is not handling it very well. It's lower frequencies do not come close to meeting the analog equivalent. Figure 3b shows the peak of the filter. It is clear that the δ transform is outperforming. If the sampling period was further reduced, the delta transform would more closely match the analog design, and the z-transform design would deviate further.



(a)



(b)

Fig. 3. (a) Elliptical filter showing comparison between analog design, z-transform design, and δ-transform design. (b) Close-up of filter peak.

VII. Conclusion

It has been shown that when the sampling time decreases and the shift operator poles tend towards $z = 1$ that the delta operator and its transform have several advantages over the shift operator and the z-transform. [6] This document has explored the delta operator and its corresponding transform and demonstrated the δ-transform's superior performance over the z-transform for both faster sampling rates and more strict finite wordlength conditions.

References

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